Wong, H. W. Willie; Tay, E. G.
Extensions of Moon's and Bagga and Beineke's theorems on dicycle and score (outdegree) structures of tournaments. (English) Zbl 08145829
Graphs Comb. 42, No. 1, Paper No. 6, 18 p. (2026).

All graphs considered are simple. There are several classical results on the dicycle (i.e., directed cycle) and score (i.e., outdegree) structures of tournaments and multipartite tournaments. In this article, the authors prove extensions of such results to graphs obtained by blowing up vertices to independent sets. More precisely, given a connected simple graph G on n vertices and a tuple $(p_1, \dots, p_n)$ of positive integers, the graph extension (or G vertex-multiplications) $G(p_1, \dots, p_n)$ is the graph whose vertex set is partitioned into independent sets $V_1, \dots, V_n$ of sizes $p_1, \dots, p_n$, respectively, with edges between $x \in V_i$ and $y \in V_j$ precisely when $ij \in E(G)$.

Let G be a graph, $(p_1, \dots, p_n)$ be a tuple of positive integers, and $S_i = (s_{i,j})_{j=1}^{p_i}$ be weakly increasing sequences of non-negative integers for $1 \leq i \leq n$. The first result states that $S_1, \dots, S_n$ form the score sequences of some orientation of $G(p_1, \dots, p_n)$ if and only if for every tuple $(k_1, \dots, k_n) \neq (p_1, \dots, p_n)$ of non-negative integers such that $k_i \leq p_i$ for all $1 \leq i \leq n$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{p_i} s_{i,j} = \sum_{\substack{u<v \\ uv \in E(G)}} p_u p_v,$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{k_i} s_{i,j} \geq \sum_{\substack{u<v \\ uv \in E(G)}} k_u k_v.$$

This is a generalization of the results of *H. G. Landau* [Bull. Math. Biophys. 15, 143–148 (1953; doi:10.1007/BF02476378)] and *J. W. Moon* [Can. Math. Bull. 5, 51–58 (1962; Zbl 0104.00804)] on score sequences of tournaments and multipartite tournaments. As a corollary, it is shown that the score lists of an orientation of a graph extension determine its strong component decomposition completely.

The next result extends a result of *K. S. Bagga* and *L. W. Beineke* [Czech. Math. J. 37(112), 323–333 (1987; Zbl 0626.05019)] on realizing the score lists of bipartite tournaments to orientations of graph extensions $G(p_1, \dots, p_n)$ under the condition that the parent graph G has a vertex r whose neighbors are mutually nonadjacent and have identical neighborhoods in G. In particular, this applies to extensions of trees by choosing r to be a leaf. The authors pose as an open problem the question of whether the result holds even when no such vertex r exists.

The next result says that a dicycle exists in an orientation D of $G(p_1, \dots, p_n)$ if and only if it contains a pure dicycle (a chordless cycle that visits each part of $G(p_1, \dots, p_n)$ at most once) or a 4-dicycle. This generalizes a theorem of *W. D. Goddard* et al. [J. Comb. Theory, Ser. B 52, No. 2, 284–300 (1991; Zbl 0736.05040)]. The authors also prove a generalisation of a theorem of *K. S. Bagga* [Some structural properties of bipartite tournaments. Purdue University (PhD Thesis) (1984)], which characterizes acyclic bipartite tournaments by the absence of 4-dicycles, to orientations D of $G(p_1, \dots, p_n)$ that contain no dicycles whose underlying partite graph is a tree. The section closes with the result that if D is an orientation of $G(p_1, \dots, p_n)$ such that $\sum_{u \in V(D)} s_u^2$ is maximum among all orientations, then D is acyclic; for tournaments this maximum characterizes the transitive tournament.

Lastly, the authors prove some results on the enumeration of acyclic multipartite tournaments. They show that acyclic n-partite tournaments are precisely those isomorphic to D_X^n, the digraph on a set $X \subset \mathbb{Z}^+$ with arcs from smaller to larger elements within each congruence class modulo n. This generalizes a result of *S. Das* et al. [Discrete Math. 344, No. 9, Article ID 112497, 17 p. (2021; Zbl 1473.05104)] for bipartite tournaments. Isomorphism of two such tournaments is characterized in four equivalent ways, including equality of score lists. Lastly, the number of nonisomorphic acyclic orientations of a complete multipartite graph with given partite sizes is computed explicitly in terms of multinomial coefficients. This generalizes a result of Das et al. [loc. cit.] for acyclic bipartite tournaments.

Reviewer: Brahadeesh Sankarnarayanan (Jodhpur)

MSC:

05C20 Directed graphs (digraphs), tournaments
05C30 Enumeration in graph theory
05C38 Paths and cycles
05C07 Vertex degrees
05C75 Structural characterization of families of graphs

Keywords:

orientation; tournament; multipartite tournament; graph extension; vertex-multiplication; score lists

Full Text: DOI

References:

[1] Bagga, K.S.: Some Structural Properties of Bipartite Tournaments, Ph.D thesis, Purdue University, (1984)

[2] Bagga, KS; Beineke, LW, Uniquely realizable score lists in bipartite tournaments, Czechoslovak Math. J., 37, 323-333, 1987 · Zbl 0626.05019 · doi:10.21136/CMJ.1987.102160

[3] Bang-Jensen, J.; Gutin, G., Generalizations of tournaments: A survey, J. Graph Theory, 28, 171-202, 1998 · Zbl 0920.05033 · doi:10.1002/(SICI)1097-0118(199808)28:4<171::AID-JGT1>3.0.CO;2-G

[4] Bang-Jensen, J.; Gutin, G., Digraphs: Theory, 2009, Algorithms and Applications: Springer Verlag, London, Algorithms and Applications · Zbl 1170.05002 · doi:10.1007/978-1-84800-998-1

[5] Beineke, L.W.: A tour through tournaments or bipartite and ordinary tournaments: A comparative survey, in Combinatorics, London Math. Soc. Lecture Note Ser., 52, Cambridge University Press, (1981), 41–55 · Zbl 0458.05036

[6] Bondy, JA, Disconnected orientation and a conjecture of Las Vergnas, J. London Math. Soc., 14, 277-282, 1976 · Zbl 0344.05124 · doi:10.1112/jlms/s2-14.2.277

[7] Das, S.; Ghosh, P.; Ghosh, S.; Sen, S., Oriented bipartite graphs and the Goldbach graph, Discrete Math., 344, 2021 · Zbl 1473.05104 · doi:10.1016/j.disc.2021.112497

[8] Gale, D., A theorem of flows in networks, Pacific J. Math., 7, 1073-1082, 1957 · Zbl 0087.16303 · doi:10.2140/pjm.1957.7.1073

[9] Goddard, WD; Kubicki, G.; Oellermann, OR; Tian, S., On multipartite tournaments, J. Combin. Theory Ser. B, 52, 284-300, 1991 · Zbl 0736.05040 · doi:10.1016/0095-8956(91)90069-V

[10] Gutin, G., Cycles and paths in semicomplete multipartite digraphs, theorems, and algorithms: A survey, J. Graph Theory, 19, 481-505, 1995 · Zbl 0839.05043 · doi:10.1002/jgt.3190190405

[11] Harary, F.; Moser, L., The theory of round robin tournaments, Amer. Math. Monthly, 73, 231-246, 1966 · Zbl 0142.41602 · doi:10.1080/00029890.1966.11970749

[12] Koh, KM; Tay, EG, On optimal orientations of G vertex-multiplications, Discrete Math., 219, 153-171, 2000 · Zbl 0946.05036 · doi:10.1016/S0012-365X(99)00373-8

[13] Landau, HG, On dominance relations and the structure of animal societies: III The condition for a score structure, Bull. Math. Biophys., 15, 143-148, 1953· doi:10.1007/BF02476378

[14] Mckay, JH, The William Lowell Putnam Mathematical Competition, Amer. Math. Monthly, 73, 726-732, 1966· doi:10.1080/00029890.1966.11970824

[15] Moon, JW, On the score sequence of an n-partite tourament, Canad. Math. Bull., 5, 51-58, 1962 · Zbl 0104.00804 · doi:10.4153/CMB-1962-008-9

[16] Moon, JW, Topics on tournaments, 1968, Rinehart and Winston, New York: Holt, Rinehart and Winston, New York · Zbl 0191.22701

[17] Naikoo, TA, On scores in tournaments, Acta Univ. Sapientiae Math., 10, 257-267, 2018 · Zbl 1421.05050 · doi:10.2478/ausi-2018-0013

[18] Reid, KB, Tournaments: Scores, kings, generalizations and special topics, Congr. Numer., 115, 171-211, 1996 · Zbl 0899.05024

[19] Reid, KB; Santana, M., Landau's theorem revisited again, J. Combin. Math. Combin. Comput., 80, 171-191, 2012 · Zbl 1242.05111

[20] Volkmann, L., Cycles in multipartite tournaments: Results and problems, Discrete Math., 245, 19-53, 2002 · Zbl 0996.05063 · doi:10.1016/S0012-365X(01)00419-8

[21] Volkmann, L., Multipartite tournaments: A survey, Discrete Math., 307, 3097-3129, 2007 · Zbl 1134.05033 · doi:10.1016/j.disc.2007.03.053

[22] Wong, WHW; Tay, EG, Arc reversals of cycles in orientations of G vertex-multiplications, Discrete Math., 345, 2022 · Zbl 1491.05096 · doi:10.1016/j.disc.2022.112999

by data from zbMATH Open. This attempts to reflect the references listed in the original paper as accurately as possible without claiming completeness or a perfect matching.